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The relation between problems of pursuit, controllability and stability in the large in linear systems with different types of constraints☆

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Abstract

A relation is established between problems of pursuit, controllability and stability in the large in linear systems when a geometric constraint is imposed on the control vector of the pursuer and an integral constraint is imposed on the control function of the evader.

Consider the following differential game[1–5]

$$dz/dt = Az - Bu + C\upsilon, \quad z \in R^n, \quad u \in R^m, \quad \upsilon \in R^l \tag{1}$$

where A, B and C are constant matrices of the corresponding dimensions. During motion, the control function of the pursuer $u(\cdot)$ must satisfy the constraint $u(t) \in P$ almost everywhere and the control function of the evader $\upsilon(\cdot)$ must satisfy the constraint

$$\int_0^\infty |\upsilon(t)|^p dp \le \sigma^p, \quad P \subset R^m, \quad \sigma \ge 0, \quad p > 1 \tag{2}$$

The aim of the pursuer and, when there is no evader (that is, when $\sigma = 0$), the aim of the control is to realize the inclusion $z(t) \in M$ at a certain t, $t > 0$, where M is a specified subset of R^n (the terminal set) and $z(t)$ is the trajectory which is the solution of Eq. (1) with the specified initial condition $z(0) = z_0$ when the actual methods of controlling the vectors u and υ are chosen by the players. We introduce the following assumptions:

1) P is a bounded subset of R^m and the origin of the coordinates is an internal point of the set P, the terminal set $\mathbf{M}$ is contained in a sphere $|x| \le \rho$, it is closed, bounded and contains the origin of the coordinates, 2) all of the Jordan boxes of matrix A, corresponding to the eigenvalues λ, for which $\mathrm{Re}\lambda = 0$, are simple and $\mathrm{rank}B = n$ (consequently, $m \ge n$).

In order to give an exact formulation of the problem, we will first define the actual strategy of the pursuer (feedback control). For this purpose, we separate out a system of linearly independent vectors $b_{j_1}, b_{j_2}, \ldots, b_{j_n}$, the columns of the matrix B, which is possible by virtue of assumption 2. We will denote the matrix, which is composed of these

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vectors as columns (in the same order), by B_*. Hence, B_* is a non-degenerate $n \times n$ matrix. Moreover, we put $J = \{j_1, j_2, \ldots, j_n\}$.

Using the vector $u \in R^m$, we next define a vector $u_J \in R^n$ by the formula $u_J = (u_{j1}, u_{j2}, \ldots, u_{jn})$ and, using the vector $w \in R^n$, we define a vector $w^J \in R^m$ by the formula $(w^J)_j = w_k$ if $j \in J, j = j_k$ and $(w^J)_j = 0$ if $j \notin J$. It is easily verified that the following relations hold

$$|w^J| = |w|, \quad (w^J)_J = w, \quad Bu = B_* u_J \tag{3}$$

We now choose a positive number ε such that a closed sphere D of radius $\varepsilon||B_*^{-1}||$ with its centre at the origin of coordinates is contained in the set P.

The strategy which has been mentioned is introduced in the following way

$$\bar{u}(z) = \varepsilon|z|^{-1}(B_*^{-1}z)^J, \text{ if } z \neq 0 \text{ и } \bar{u}(0) = 0 \tag{4}$$

We denote the set of all measurable functions $u(\cdot)$:$[0; +\infty) \to P$ (the functions $\upsilon(\cdot) : [0; +\infty) \to R^l$ respectively) which satisfy the constraint (2)) by U (V respectively). Suppose an arbitrary control $\upsilon(\cdot) \in V$ is chosen by the evader. Then, the trajectory, generated by the initial state z_0, by the strategy $\bar{u}(z)$ and by the function $\upsilon(\cdot)$ is defined as the solution according to Filippov[6] of the Cauchy problem

$$dz/dt = Az - B\bar{u}(z) + C\upsilon(t), \quad z(0) = z_0 \tag{5}$$

which has a unique and absolutely continuous solution $z(t)$, which is extendable on the semi-axis $[0, +\infty)$.

By definition, the pursuit can be completed from the point z_0 in the game (1), if, for each function $\upsilon(\cdot) \in V$, the trajectory $z(t)$, which is generated by z_0, $\bar{u}(t)$ and $\upsilon(\cdot)$, satisfies the condition $z(t) \in M$ at a certain t, $t > 0$.

We recall that the system

$$dz/dt = Az - Bu, \quad u \in P \tag{6}$$

is designated as M-controllable in the large if a control function $u(\cdot) \in U$ exists for any initial point $z_0 \in R^n$ such that the inclusion $z(t) \in M$ holds at a certain t, $t \geq 0$ in the case of the solution $z(t)$ of system (6) with the condition $z(0) = z_0$.[7,8]

Theorem. *Suppose assumptions 1 and 2 are satisfied. Then, the following assertions are equivalent:*

a) *in the game* (1), *the pursuit can be completed from any point* z_0;
b) *system* (6) *is M-controllable in the large;*
c) *the system* $d\dot{z}/dt = Az$ *is Lyapunov stable.*

Proof.

1°. Assuming that assertion *a* holds, we will derive assertion *b*. In system (5), we put $\upsilon(t) \equiv 0$, $t \geq 0$. Then, the inclusion $z(\hat{t}) \in M$ must hold for the corresponding trajectory in the case of a certain $\hat{t}$, $\hat{t} > 0$. By virtue of the closedness of the set M, it can be assumed that $z(t) \neq 0$ when $0 \leq t < \hat{t}$. We now define a control function $\hat{u}(t)$, which is called the realization of the strategy $\bar{u}(z)$, in the following manner: if $z(\hat{t}) \neq 0$, then $\hat{u}(t) = \bar{u}(z(t))$ when $0 \leq t \leq \hat{t}$ and $\hat{u}(t) = 0$ when $t > \hat{t}$. If, however, $z(\hat{t}) = 0$, then $\hat{u}(t) = \bar{u}(z(t))$ when $0 \leq t < \hat{t}$ and $\hat{u}(t) = 0$ when $t \geq \hat{t}$. In both cases, the solution of the Cauchy problem $\dot{z} = Az - B\hat{u}(t)$, $z(0) = z_0$ is identical to the function $z(t)$ and $z(\hat{t}) \in M$. By virtue of the arbitrariness of z_0, assertion *b* is thereby proved.□

Remark. The function $\hat{u}(z)$, which is a strategy in the differential game, corresponds to the synthesis of a control (a feedback control or a control along a closed contour) in a control problem concerning the translation of a phase point from an initial state into the set of states M. Assertion *b* of the theorem can therefore be reformulated as follows: in system (6), a synthesis of controls exists in the whole space. In this formulation, the implication $a \Rightarrow$ b is almost obvious.

2°. We will now prove that assertion *c* follows from assertion *b*. We will assume that the opposite is true: the system $dz/dt = Az$ is unstable. We shall show that system (6) is then not M-controllable as a whole.

Under the conditions of the theorem, it follows from the instability of the system $dt/dt = Az$ that the matrix A has just a single eigenvalue λ such that $\mathrm{Re}\lambda > 0$.[9]

Two cases are possible. First case: λ is a real number. A real vector $h \in R^n$ then exists such that $|h| = 1$, $A^*h = \lambda h$, where A^* is the transpose of matrix A. We choose $z_0 = \gamma h$ as the initial point (the positive factor γ will be chosen later). Suppose $u(\cdot) \in U$ is an arbitrary control function and $z(t)$ is the corresponding trajectory of system (6). We put

$$\xi(t) = h \cdot z(t), \quad w(t) = h \cdot Bu(t)$$

(a scalar product is denoted by a dot). Then,

$$\dot{\xi}(t) = \lambda\xi(t) - w(t), \quad \xi(0) = \gamma$$

Hence,

$$|\xi(t)| = \left| e^{\lambda t}\gamma - \int_0^t e^{\lambda(t-s)} w(s)ds \right| \geq \gamma e^{\lambda t} - \int_0^t e^{\lambda(t-s)}|w(s)|ds \geq \gamma e^{\lambda t} - R\frac{e^{\lambda t} - 1}{\lambda}, \quad R = \max_{u \in P}(h \cdot Bu)$$

Consequently, if $\gamma > \max\{R/\lambda, \rho\}$, then

$$|z(t)| \geq |\xi(t)| \geq e^{\lambda t}(\gamma - R/\lambda) + R/\lambda > \rho \tag{7}$$

Inequality (7) shows that $x(t) \bar{\in} M$ for all t, $t > 0$, that is, system (7) is not M-controllable.

We will now consider the second case when the eigenvalue λ is complex, $\lambda = \alpha + i\beta$, $\alpha > 0$, $\beta \neq 0$. Suppose $h = h_1 + ih_2$ is the corresponding eigenvector. We have

$$A^*h_1 = \alpha h_1 - \beta h_2, \quad A^*h_2 = \beta h_1 - \alpha h_2$$

so that $h_1 \neq 0$ and $h_2 \neq 0$.[9] We put

$$\xi_i = h_i \cdot z, \quad w_i(t) = h_i \cdot Bu(t), \quad \xi_{i_0} = h_i \cdot z_0, \quad i = 1, 2 \tag{8}$$

Then, the vector $\xi = (\xi_1, \xi_2)$ satisfies the system

$$\frac{d\xi}{dt} = \begin{Vmatrix} \alpha & -\beta \\ \beta & \alpha \end{Vmatrix} \xi - \begin{Vmatrix} w_1(t) \\ w_2(t) \end{Vmatrix} \tag{9}$$

Taking account of relations (8) and (9), as in the first case, we derive the limit

$$|z(t)| \geq \frac{h_1}{|h_1|} \cdot z = \frac{\xi_1}{|h_1|} \geq \frac{1}{\chi}\left[e^{\alpha t}\alpha(t) - \int_0^t e^{\alpha(t-s)}|w_1(s)\cos\beta s - w_2(s)\sin\beta s| ds \right] \tag{10}$$

where

$$a(t) = |\xi_{10}\cos t - \xi_{20}\sin\beta t|$$

and the number χ is chosen from the condition

$$\chi R \geq \max_{u \in P}|h_i| \cdot |u(t)|, \quad i = 1, 2$$

Since $|w_i(t)| \leq |h_i| \cdot |u(t)| \leq \chi R$, it follows from this that

$$|z(t)| \geq \frac{a(t)}{\chi}e^{\alpha t} - \frac{2}{\alpha}R(e^{\alpha t} - 1) \tag{11}$$

Similarly,

$$|z(t)| \geq \frac{b(t)}{\chi}e^{\alpha t} - \frac{2}{\alpha}R(e^{\alpha t} - 1); \quad b(t) = |\xi_{10}\sin\beta t - \xi_{20}\cos\beta t| \tag{12}$$

The inequality

$$|z(t)| \geq \frac{a(t)+b(t)}{2\chi}e^{\alpha t} - \frac{2}{\alpha}R(e^{\alpha t}-1)$$

follows from relations (11) and (12).

Note that

$$a(t)+b(t) \geq \sqrt{a(t^2)+b(t^2)} = \sqrt{\xi_{10}^2+\xi_{20}^2} = |\xi_0|$$

Hence, if $|\xi_0| > 4\chi R/\alpha$, then

$$|z(t)| \geq \left[\frac{|\xi_0|}{2\chi} - \frac{2}{\alpha}R\right]e^{\alpha t} + \frac{2}{\alpha}R \geq \frac{|\xi_0|}{2\chi}$$

Consequently, if z_0 is chosen such that the inequality

$$|\xi_0| > \max\{4\chi R/\alpha, 2\chi\rho\}$$

is satisfied, then $|z(t)| > \rho$, $t \geq 0$, that is, $z(t) \bar{\notin} M$ for any t, $t \geq 0$.

We therefore arrive at a contradiction in both cases.

3°. We will now prove that assertion *a* follows from assertion *c*. Suppose the pursuer adheres to strategy (4), $v(\cdot) \in V$ is the arbitrary control of the evader, z_0 is an arbitrary initial point ($z_0 \neq 0$) and $z(t)$ is the corresponding solution of system (1). We will estimate the norm of the vector $z(t)$:

$$\frac{d|z|}{dt} = \frac{z}{|z|}\cdot\dot{z} = \frac{1}{|z|}z\cdot Az - \varepsilon + \frac{z}{|z|}\cdot Cv(t)$$

Since the system $\dot{z} = Az$ is stable, it is necessary that $z \cdot Az \leq 0$ for all $z \in R^n$ (Ref. 9) and, therefore,

$$\frac{d}{dt}|z(t)| \leq -\varepsilon + \|C\|\cdot|v(t)|$$

Integrating the last equality, we obtain

$$\begin{aligned} |z(t)| &\leq |z_0| - \varepsilon t + \|C\|\cdot\int_0^t |v(t)|dt \leq |z_0| - \varepsilon t + \\ &+ \|C\|\cdot\left(\int_0^t 1^p dt \cdot \int_0^t |v(t)|^p dt\right)^{1/p} \leq |z_0| - \varepsilon t + \|C\|\cdot\sigma t^{1/p} \end{aligned} \quad (13)$$

Since $p > 1$, it follows from inequality (13) that $z(t) = 0 \in M$ for a certain t, $t \leq T(z_0)$, where $T(z_0)$ is the first positive root of the equation

$$|z_0| - \varepsilon t + \sigma\|C\|t^{1/p} = 0$$

Remark. Assertions similar to the theorem which has been proved are also true in cases when the constraints imposed on the control vectors for the two players are simultaneously geometric or simultaneously integral, which is far easier to prove. As regards the case when the control vector of the pursuer satisfies an integral constraint and that of the evader satisfies a geometric constraint, then, generally speaking, there is no analog of the theorem which has been proved.

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